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On the terminal condition for the Bellman equation for dynamic optimization with an infinite horizon

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ABSTRACT

In this work a sufficient condition for deterministic dynamic optimization with discrete time and infinite horizon is formulated. It encompasses also situations where the instantaneous payoff/utility function can attain infinite values.

The usual terminal condition for sufficiency of the Bellman equation requiring that the limit superior of the value function along each admissible trajectory is equal to 0 is replaced by a weaker one in which the limit superior of the value function can attain nonpositive values.

This kind of terminal condition is applicable also to deterministic dynamic optimization problems with real-valued instantaneous payoff function in which the usual terminal condition does not hold.

1. Introduction

Since Bellman's seminal book [1], the theory of dynamic programming has evolved in many directions, increasing the class of problems that it is able to cope with. Now it is, in both its deterministic and stochastic versions, standard textbook material (see e.g. Feinberg and Shwartz [2], Judd [3] or Stokey et al. [4]).

This work continues on the deterministic case, along the lines of, among other studies, Blackwell [5], Strauch [6] or Stokey et al. [4].

In the papers of Blackwell [5] and Stokey et al. [4] the infinite horizon discounted dynamic optimization problems were considered, while Strauch [6] considered also the infinite horizon undiscounted dynamic optimization problems with nonpositive instantaneous payoff function.

There was, among other features, a sufficient condition that a function obtained in some way is actually the value function of the problem, consisting of the Bellman equation with the terminal condition that the discounted value of this function along every admissible trajectory has limit superior equal to 0. We shall refer to this as to the *traditional terminal condition*. Precise formulation of a satisfactory condition using the Bellman equation with a new terminal condition appears in Theorem 1, while the traditional terminal condition appears in Remark 1.

This kind of terminal condition sometimes does not hold even in the case where the instantaneous payoff function is real valued – such a simple counterexample is formulated in Section 3.2. Moreover, it usually does not hold in the case where we allow the instantaneous payoff to reach $-\infty$, as is assumed in this work.

Nevertheless, the results can be applied also to the case of real-valued instantaneous payoff functions and they work in the cases where the existing theorems cannot be applied.

Such dynamic optimization problems appear in some mathematical models of exploitation of renewable resources or problems with pollution – it may be reasonable to consider some socially disastrous states of the system or the resulting

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situation of its users by assigning to them a payoff equal to $-\infty$. Such models were considered by, among others, Levhari and Mirman [7], and Fisher and Mirman [8], and in some papers by the author [9–12].

Lack of an appropriate terminal condition is a real problem – in some of the papers devoted to such applications (e.g. Levhari and Mirman [7] and Fisher and Mirman [8]) a candidate for a value function is found and checking optimality is restricted to checking the Bellman equation only – without checking any kind of terminal condition. Such a procedure may lead to erroneous results. Moreover, the traditional terminal condition does not hold in those models. Nevertheless, the calculations contained there can become correct if we use the terminal condition introduced in this work.

2. Formulation of the problem

We start with a formal introduction of notation.

We consider the problem of dynamic maximization with *discrete time* and *infinite horizon*; therefore, without loss of generality, we number the consecutive time instants by elements of the set of nonnegative integers $\mathbb{N}$.

Such a problem can be regarded as the sextuple

$$\mathcal{M} = (\mathbb{X}, \mathbb{U}, g, \beta, f, D)$$

defined below.

The *set of state variables* will be denoted by $\mathbb{X}$ and the set of *control parameters* by $\mathbb{U}$.

The *instantaneous payoff function* is $g : \mathbb{X} \times \mathbb{U} \times \mathbb{N} \to \mathbb{R} \cup \{-\infty\}$ with the *discount factor* $0 < \beta \leq 1$.

The function describing *change of the state variable* is $f : \mathbb{X} \times \mathbb{U} \times \mathbb{N} \to \mathbb{X}$ and the correspondence describing *availability of controls* is $D : \mathbb{X} \times \mathbb{N} \multimap \mathbb{U}$ with nonempty values.

Any function $U : \mathbb{X} \times \mathbb{N} \to \mathbb{U}$ fulfilling $U(x, t) \in D(x, t)$ for every $x \in \mathbb{X}$ and $t \in \mathbb{N}$ will be called an *admissible (closed loop) control function* and the set of all *admissible control functions* will be denoted by $\mathcal{U}$.

Besides the dynamic optimization problem $\mathcal{M}$ we have an *initial state* $\bar{x} \in \mathbb{X}$.

Given $U \in \mathcal{U}$, any function $X : \mathbb{N} \to \mathbb{X}$ given recurrently by the condition $X(0) = \bar{x}$ and $X(t+1) = f(X(t), U(X(t), t), t)$, for every t, is called the *trajectory* (of the state variable) *corresponding to* U and $\bar{x}$.

Given $\mathcal{M}$ and $\bar{x}$, our problem is to maximize over $U \in \mathcal{U}$ the function

$$J(\bar{x}, U) = \sum_{t=0}^{+\infty} g(X(t), U(X(t), t), t) \cdot \beta^t$$

with X being the trajectory corresponding to U and $\bar{x}$.

Formally, we say that $\bar{U}$ is an *optimal control function* for $\mathcal{M}$ and $\bar{x}$ if

$$J(\bar{x}, \bar{U}) = \sup_{U \in \mathcal{U}} J(\bar{x}, U).$$

This will be denoted by $\bar{U} \in \mathrm{Sol}(\mathcal{M}, \bar{x})$.

We shall consider only problems such that $J : \mathbb{X} \times \mathcal{U} \to \overline{\mathbb{R}}$ (where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$) is always well defined.

The class of such problems encompasses, among other cases, ones where $\beta < 1$ and g is bounded from above (containing the discounted case of Blackwell [5]), as well as cases where g is nonpositive (containing the negative case of Strauch [6]).

In this work there is no need to restrict to such narrow classes of problems.

Theorem 1. *Assume that a function* $V : \mathbb{X} \times \mathbb{N} \to \overline{\mathbb{R}}$ *fulfills the condition (the* Bellman equation*)*

$$V(x, t) = \sup_{u \in D(x,t)} g(x, u, t) + \beta \cdot V(f(x, u, t), t + 1)$$

with the following terminal condition*: for every trajectory X,*

$$\limsup_{t\to\infty} V(X(t), t) \cdot \beta^t \leq 0$$

$$\textit{and if } \limsup_{t\to\infty} V(X(t), t) \cdot \beta^t < 0 \quad \textit{then } J(\bar{x}, U) = -\infty$$

for every trajectory U such that X is corresponding to U and $\bar{x}$.

Then:

(a) *The function V is the value function of the dynamic optimization problem, i.e.*

$$V(\bar{x}, 0) = \sup_{U \in \mathcal{U}} J(\bar{x}, U).$$

(b) *If, moreover, a control function $\bar{U}$ fulfills*

$$\bar{U}(x, t) \in \operatorname*{Argmax}_{u \in D(x,t)} g(x, u, t) + \beta \cdot V(f(x, u, t), t + 1) \quad \textit{for every } x \in \mathbb{X},\ t \in \mathbb{N},$$

then

$$\bar{U} \in \mathrm{Sol}(\mathcal{M}, \bar{x}).$$

Remark 1. The *traditional terminal condition* is

$$\limsup_{t\to\infty} V(X(t),t)\cdot\beta^t = 0.$$

Proof of Theorem 1. (a)(i) First we prove that

$$V(\bar{x},0) \geq \sup_{U\in\mathcal{U}} J(\bar{x},U).$$

If $J(\bar{x},U) = -\infty$, then the inequality $V(\bar{x},0) \geq J(\bar{x},U)$ is obviously fulfilled.
In any other case we have $\limsup_{t\to\infty} V(X(t),t)\cdot\beta^t = 0$.
By the Bellman equation, for every control function U and the trajectory X corresponding to U and $\bar{x}$ we have

$$V(X(t),t) \geq g(X(t),U(X(t),t),t) + \beta\cdot V(f(X(t),U(X(t),t)),t+1).$$

If we repeat this procedure for n consecutive time instants, starting from $t=0$, we obtain

$$V(\bar{x},0) = V(X(0),0) \geq \sum_{t=0}^{n} g(X(t),U(X(t),t),t)\cdot\beta^t + \beta^{n+1} V\left(X(n+1),n+1\right).$$

Since $\limsup_{t\to\infty} V(X(t),t)\cdot\beta^t = 0$, then for every $\varepsilon > 0$ there exists $N\in\mathbb{N}$ and a subsequence such that $\beta^{n_k}\cdot V(X(n_k),n_k) > -\frac{\varepsilon}{2}$.
On the other hand, J is well defined. So either it is infinite or there exists $N_1 \geq N$ such that for every $n_k > N_1$

$$\sum_{t=0}^{n_k} g(X(t),U(X(t),t),t)\cdot\beta^t \geq J(\bar{x},U) - \frac{\varepsilon}{2}.$$

So for finite $J(\bar{x},U)$ and $\limsup_{t\to\infty} V(X(t),t)\cdot\beta^t = 0$ we have $V(\bar{x},0) \geq J(\bar{x},U)-\varepsilon$, which implies $V(\bar{x},0)\geq J(\bar{x},U)\cdot$.
If $J(\bar{x},U) = +\infty$, then for every $M > 0$ there exists $N_1 \geq N$ such that for every $n_k > N_1$ we have

$$\sum_{t=0}^{n_k} g(X(t),U(X(t),t),t)\cdot\beta^t > M + \frac{\varepsilon}{2}.$$

This implies $V(\bar{x},0) \geq M$ for every M, i.e. $V(\bar{x},0) = +\infty$, which guarantees that the inequality $V(\bar{x},0) \geq J(\bar{x},U)$ is fulfilled.
(ii) Now we prove that

$$V(\bar{x},0) \leq \sup_{U\in\mathcal{U}} J(\bar{x},U).$$

Let us take any $\delta > 0$.
The Bellman equation also implies that for every t and x there exists $u_{x,t}$ such that

$$V(x,t) \leq g(x,u_{x,t},t) + \beta\cdot V(f(x,u_{x,t},t),t+1) + \frac{\delta}{2^t}.$$

We define a control function U by $U(x,t) = u_{x,t}$.
For this U and X corresponding to U and $\bar{x}$ we have

$$V(X(t),t) \leq g(X(t),U(X(t),t),t) + \beta\cdot V(f(X(t),U(X(t),t)),t+1) + \frac{\delta}{2^t}.$$

If we repeat this procedure for n consecutive time instants, starting from $t=0$, we obtain

$$\begin{aligned} V(\bar{x},0) &= V(X(0),0) \\ &\leq \sum_{t=0}^{n} g(X(t),U(X(t),t),t)\cdot\beta^t + \beta^{n+1}\cdot V(X(n+1),n+1) + \sum_{t=0}^{n}\frac{\delta}{2^t}\cdot\beta^t \\ &\leq \sum_{t=0}^{n} g(X(t),U(X(t),t),t)\cdot\beta^t + \beta^{n+1}\cdot V(X(n+1),n+1) + \frac{2\delta}{2-\beta}. \end{aligned}$$

Now let us take any $\varepsilon > 0$. If δ introduced above fulfills $\delta < \frac{\varepsilon\cdot(2-\beta)}{6}$, then $\frac{2\delta}{2-\beta} < \frac{\varepsilon}{3}$.
Since $\limsup_{t\to\infty} V(X(t),t)\cdot\beta^t \leq 0$, there exists N such that for every $n > N$ $\beta^n\cdot V(X(n),n) < \frac{\varepsilon}{3}$.
Since J is well defined, then it is either infinite or finite.
In the latter case there exists $N_1 > N$ such that for every $n > N_1$

$$\sum_{t=0}^{n} g(X(t),U(X(t),t),t)\cdot\beta^t < J(\bar{x},U) + \frac{\varepsilon}{3}.$$

In this case we have

$$V(\bar{x},0) \le J(\bar{x},U) + \varepsilon.$$

If $J(\bar{x},U) = -\infty$, then for every $M < 0$ there exists $N_1 > N$ such that for every $n > N_1$

$$\sum_{t=0}^{n} g(X(t),U(X(t),t),t) \cdot \beta^t < M + \frac{2\varepsilon}{3}.$$

Then we have $V(\bar{x},0) \le M$, which implies $V(\bar{x},0) = -\infty$.

If $J(\bar{x},U) = +\infty$, then the inequality is obviously fulfilled.

(b) Note that for $\bar{U}$ fulfilling

$$\bar{U}(x,t) \in \operatorname*{Argmax}_{u \in D(x,t)} g(x,u,t) + \beta \cdot V(f(x,u,t),t+1)$$

and X being the trajectory corresponding to $\bar{U}$ and $\bar{x}$, the inequalities describing relations between $V(X(t),t)$ and

$$g(X(t),\bar{U}(X(t),t),t) + \beta \cdot V(f(X(t),\bar{U}(X(t),t)),t+1)$$

in the proofs of (a)(i) and (ii) are replaced by the equality

$$V(X(t),t) = g(X(t),U(X(t),t),t) + \beta \cdot V(f(X(t),U(X(t),t)),t+1).$$

So, repeating reasoning analogous to that in (a)(i) and (ii), we obtain

$$V(\bar{x},0) = \sum_{t=0}^{n} g(X(t),\bar{U}(X(t),t),t) \cdot \beta^t + \beta^{n+1} \cdot V(X(n+1),n+1),$$

while all the remaining constraints hold for this $\bar{U}$.

This implies that $V(\bar{x},0) = J(\bar{x},\bar{U}) = \sup_{U \in \mathcal{U}} J(\bar{x},U)$. □

3. An example

3.1. A common ecosystem

Let us consider a dynamic optimization model derived from a model of a common ecosystem examined by, among others, Wiszniewska-Matyszkiel [12] in a game theoretic model with many players illustrating the so called "tragedy of the commons".

In the original example there were many players, each of them exploiting the same renewable resource – rainforest – and maximizing profits from extraction over an infinite time horizon. Here we consider a version with only one player.

Here $\mathbb{X} = \mathbb{U} = \mathbb{R}_+$ – the set of nonnegative reals.

The function describing the behaviour of the ecosystem is $f(x,u,t) = x - \max(0, u - \zeta x)$ for $\zeta > 0$, called the *regeneration rate*.

The sets of available control parameters are $D(x,t) = [0,(1+\zeta) \cdot x]$.

The instantaneous payoff function is $g(x,u,t) = \ln u$, with $\ln 0$ understood as $-\infty$. An instantaneous payoff independent of the state variable corresponds to a situation where the costs of extraction are independent of the state variable and the revenues depend, as in real life, only on the amount extracted.

Similar examples, in which the instantaneous payoff function was directly independent of the state variable and the dependence of the payoff on it was via the sets of available control parameters, were considered by e.g. Levhari and Mirman [7], and Fisher and Mirman [8].

The discount factor is $\beta = \frac{1}{1+r}$ for some $r > 0$.

This completes the definition of $\mathcal{M}$.

We assume that the *initial state* $\bar{x} > 0$.

In this model obviously we have trajectories with $X(t) = 0$ for some t and, therefore, $\limsup_{t\to\infty} V(X(t),t) \cdot \beta^t = -\infty$.

Therefore, the traditional terminal condition cannot be used, but Theorem 1 can be applied.

Proposition 2. *The value function for $\mathcal{M}$ is defined by*

$$V(x,t) = A \ln x + B$$

for

$$A = \frac{1+r}{r} > 0$$

and

$$B = \begin{cases} \frac{1+r}{r} \cdot \ln \zeta & \textit{if } r \leq \zeta, \\ \frac{1+r}{r} \cdot \ln \left(\frac{r \cdot (1+\zeta)}{1+r} \right) + \ln \left(\frac{1+\zeta}{1+r} \right) \cdot \frac{1+r}{r^2} & \textit{otherwise;} \end{cases}$$

A control function defined by $U(x,t) = c \cdot x$ *for*

$$c = \begin{cases} \zeta & \textit{if } r \leq \zeta, \\ \frac{r \cdot (1+\zeta)}{1+r} & \textit{otherwise} \end{cases}$$

belongs to $\mathrm{Sol}(\mathcal{M}, \bar{x})$.

Proof. We can do this in two ways.

First of all we can assume that $U(x,t) = c \cdot x$ and find c maximizing $J(\bar{x}, U)$ over the set of U of this form; then we can substitute the optimal payoff into the Bellman equation and check it.

The other way to proceed is to assume such a form of V, and find the constants from the Bellman equation and, finally, U.

Whatever way we choose, we obtain

$$A = \frac{1+r}{r}, \quad \text{which is strictly positive,}$$

$$B = \begin{cases} \frac{1+r}{r} \cdot \ln \zeta & \text{if } r \leq \zeta, \\ \frac{1+r}{r} \cdot \ln \left(\frac{r \cdot (1+\zeta)}{1+r} \right) + \ln \left(\frac{1+\zeta}{1+r} \right) \cdot \frac{1+r}{r^2} & \text{otherwise;} \end{cases}$$

and

$$c = \begin{cases} \zeta & \text{if } r \leq \zeta, \\ \frac{r \cdot (1+\zeta)}{1+r} & \text{otherwise.} \end{cases}$$

Checking this is an obvious but slightly dull calculation.

What remained to be proven was the terminal condition.

Let U be an admissible control function and X the trajectory corresponding to U and $\bar{x}$.

Since $X(t) \leq \bar{x}$ for every t we have the first part of the terminal condition

$$\begin{aligned} \limsup_{t \to \infty} V(X(t), t) \cdot \beta^t &\leq \limsup_{t \to \infty} V(\bar{x}, t) \cdot \beta^t \\ &= \limsup_{t \to \infty} (A \ln \bar{x} + B) \cdot \beta^t = 0. \end{aligned}$$

Now let us assume that $\limsup_{t \to \infty} V(X(t), t) \cdot \beta^t < 0$.

Since for every t we have $U(X(t), t) \leq (1+\zeta) \cdot X(t)$, we obtain

$$\sum_{t=0}^{\infty} \ln (U(X(t), t)) \cdot \beta^t \leq \sum_{t=0}^{\infty} (\ln(1+\zeta) + \ln X(t)) \cdot \beta^t.$$

This equals

$$\sum_{t=0}^{\infty} \ln(1+\zeta) \cdot \beta^t + \sum_{t=0}^{\infty} \ln X(t) \cdot \beta^t,$$

since the first sum is finite.

We substitute $\ln X(t) \cdot \beta^t = \frac{V(X(t),t)-B}{A} \cdot \beta^t$, which implies

$$\limsup_{t \to \infty} \ln X(t) \cdot \beta^t = \limsup_{t \to \infty} \frac{V(X(t), t) - B}{A} \cdot \beta^t < 0.$$

Therefore, the second series is convergent to $-\infty$, which implies that $J(\bar{x}, U) = -\infty$. $\square$

3.2. A counterexample

It may seem that in order to avoid problems with infinite value functions along some trajectories it is enough to restrict the set of available control parameters such that g can attain only finite values. In our example this can be obtained by changing the sets of state variables and control parameters to $\widetilde{\mathbb{X}} = \widetilde{\mathbb{U}} = (0, +\infty)$ and the correspondence D to $\widetilde{D}(x, t) = (0, (1 + \zeta) \cdot x)$.

This defines a new problem $\widetilde{\mathcal{M}}$

Note that such a modification does not change the optimal control:

$$\mathrm{Sol}(\mathcal{M}, \bar{x}) = \mathrm{Sol}(\widetilde{\mathcal{M}}, \bar{x}).$$

So we have a problem analogous to the previous one but with the instantaneous payoff always finite.

Therefore, we can try using the traditional terminal condition for sufficiency of the Bellman equation $\limsup_{t\to\infty} V(X(t), t) \cdot \beta^t = 0$.

Unfortunately, it turns out that it also does not work in this restricted case.

Proposition 3. *Consider the problem* $\widetilde{\mathcal{M}}$.

For every $\bar{x} > 0$ *there exists a control function* U *such that for the trajectory* X *corresponding to* U *and* $\bar{x}$

$$\limsup_{t\to\infty} V(X(t), t) \cdot \beta^t < 0.$$

Proof. An example of such a control function is

$$U(t, x) = \left(1 + \zeta - \varepsilon^{\frac{1}{\beta^t}}\right) \cdot x \quad \text{for some } 0 < \varepsilon < 1.$$

For the trajectory X corresponding to U and $\bar{x}$,

$$X(t + 1) = X(t) \cdot \varepsilon^{\frac{1}{\beta^t}}.$$

As we have proven in Proposition 2, $V(x, t) = A \ln x + B$ with A strictly positive, and both constants as calculated in Proposition 2.

This implies

$$\begin{aligned} V(X(t), t) \cdot \beta^t &= (A \cdot \ln X(t) + B) \cdot \beta^t = \left(A \cdot \ln\left(X(t-1) \cdot \varepsilon^{\frac{1}{\beta^{t-1}}}\right) + B\right) \cdot \beta^t \\ &= \left(A \cdot \left(\ln X(t-1) + \frac{1}{\beta^{t-1}} \ln \varepsilon\right) + B\right) \cdot \beta^t = A \cdot \ln X(t-1) \cdot \beta^t + B \cdot \beta^t \\ &\quad + A \cdot \beta \cdot \ln \varepsilon \leq (A \ln \bar{x} + B) \cdot \beta^t + A \cdot \beta \cdot \ln \varepsilon. \end{aligned}$$

Therefore,

$$\limsup_{t\to\infty} V(X(t), t) \cdot \beta^t \leq \limsup_{t\to\infty} (A \ln \bar{x} + B) \cdot \beta^t + A \cdot \beta \cdot \ln \varepsilon < 0. \quad \square$$

Proposition 3 shows that even in the case where the instantaneous payoff can have only finite values, the traditional terminal condition may be inapplicable – where in the problem $\widetilde{\mathcal{M}}$ we cannot prove that a candidate for the value function fulfilling the Bellman equation is really a value function without the terminal condition from this work.

4. Conclusions

As was stated in Section 1, in some dynamic optimization problems implied by applications, especially concerning ecological problems, the instantaneous payoff function at some suboptimal control functions may attain $-\infty$, which represents situations regarded as extremely disastrous.

In such problems the well known sufficient condition using the Bellman equation and the traditional terminal condition cannot be applied. For such problems a weaker form of the sufficient condition was designed, allowing both the instantaneous payoff to attain $-\infty$ at some points and, what is more important, a new form of the terminal condition.

Although the problem of infinite values of the instantaneous payoff at some boundary points can be solved by imposing additional assumptions which exclude such points without changing the optimal control, as was done in Section 3.2, the same procedure does not solve the problem of the fact that the traditional terminal condition does not hold even in the modified problem.

Therefore, the new sufficient condition using the Bellman equation and a new form of the terminal condition is also very important for solving dynamic optimization problems with real-valued instantaneous payoffs.

Similar problems may appear in dynamic optimization problems with continuous time, as well as stochastic dynamic optimization problems. Hence, these two directions are obvious routes for further research.

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